Optimal control applied for economic stabilization

Part 2

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From condition(19)we have : $p^* - p(0)e^{at}$

$$(31)$$

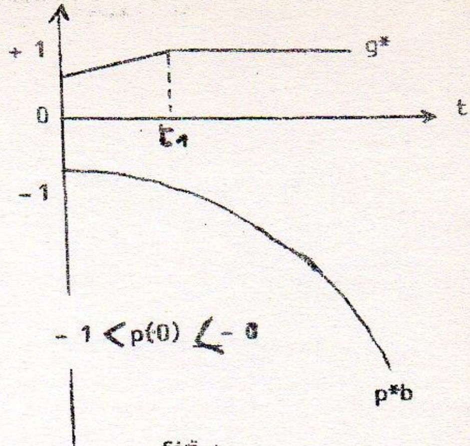
Which gives us the possible situations as given below:(see also fig.1,2,3,4).

$g^* = 1$ if $p(0) \leq -1$ for all t (32)

$$g^* = \begin{cases} -p^*b \text{ for } -1 < p(0) < 0 \text{ and } T < t_1 \text{ and } t \in [0, t_1] & (33) \\ \text{OR} \\ 1 \text{ for } t \in [t_1, T] \quad \text{for } -1 < p(0) < 0 \text{ and } T \geq t_1 \\ \underline{\text{and}} - p^*b \text{ for } t \in [0, t_1] \end{cases}$$

$$g^* = \begin{cases} -p^*b \text{ for } 0 < p(0) < 1 \text{ and } T < t_2 \text{ and } t \in [0, t_2] & (34) \\ \text{OR} \\ -p^*b \text{ for } t \in [0, t_2] \quad \text{for } 0 < p(0) < 1 \text{ and } T \geq t_2 \\ \underline{\text{and}} - 1 \text{ for } t \in [t_2, T] \end{cases}$$

$g^* = -1$ for $p(0) \geq 1$ for all $t \in [0, T]$ (35)



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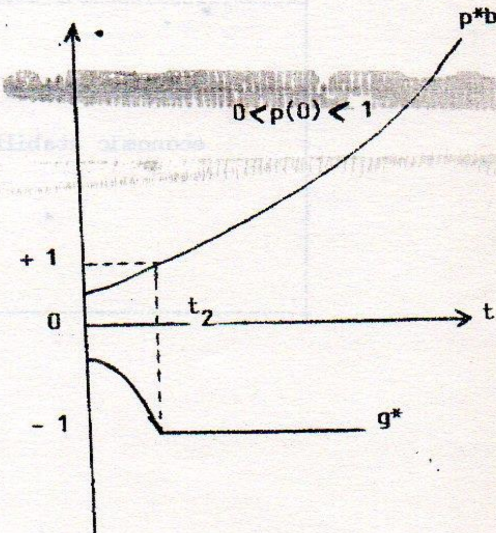
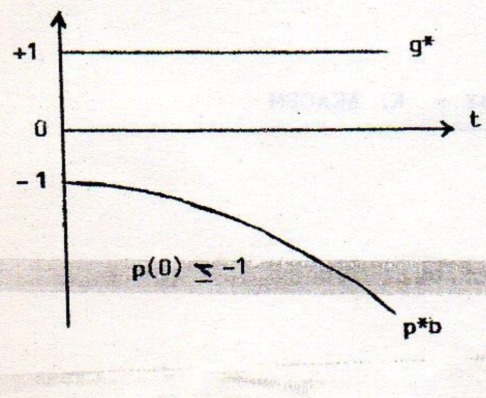


fig . 3



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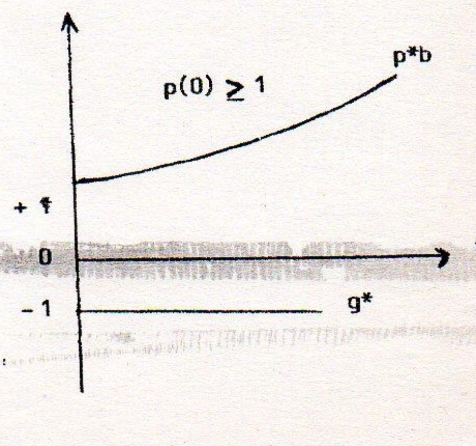

fig . 4

Where :

for the set (33), t_1 is such that :

$$p^*(t_1)\,b = -1 \qquad (36)$$

$$p^*(t)\,b < -1 \qquad \text{for } t > t_1$$

and for the set (34), t_2 is such that

$$p^*(t_2)\,b = 1 \qquad (37)$$

$$p^*(t)\,b > 1 \qquad \text{for } t > t_2$$

the case $g^*=0$ gives $y^*(t)=y_0 e^{-at}$ so that $y^*(T)=0$ requires $T=\infty$, Which we exclude .

Notice that statements (33)and (34) would eventually give policies in two phases that would result in some control expenditure saving . INdeed in these cases the control variable magnitude reaches its highest possible values(-1 or -1) only after a certain time, and for a part only, of the stabilization time , unlike the bang bang-bang stabilization policy where the policy variable is equal to one (or the other) of its constraint boundaries during all the stabilization horizon. We already get here some feeling about the importance of the timing of a policy in économic stabilization. Indeed statements (33) and (34) would tell us <u>when</u> and for <u>how long</u> which policy should be used.

d . The Optimal Policy :

In this paragraph we will try to see which policy, among the possible ones, is optimal for each possible situation of the initial state y(0) (positive or neagative) In other words we shall see which policy (or policies) would transfer positive initial states to the origin and which would transfer negative initial states of the économy . We shall do this inthe following way :

d . 1 Case : $g^* = 1$

When $g^* = 1$

$$y^*(t) = y_0 e^{-at} + b(1-e^{-at}) \qquad (38)$$

which vanishes only when $y_0 < 0$ since $b > 0$.

But $y_0 < 0$ implies y(t) negative which tells us that the control $g^*(t) = 1$ can do the job only when $y(t) < 0$.

d . 2. Case: $g^* = -1$

Similarly , it can be shown that the policy $g^* = -1$ can transfer only positive(initial)states of the économy to the origin .

d.3. Case: $g^* = -p^*b$, $T < t$ or $t \in (0,T)$

As can easily be shown, this policy can do the job only for y(t) negative . When y(0) is positive, the policy

$g^*=-p^*b, T<t_2$ can be used .

d. 4 . Case :

$$g^*=\begin{cases} -p^*b \text{ for } t\in(o,t_1) \text{ and for } -1<p(o)< o,\ T\geq t_1 \\ 1 \quad \text{for } t\in[t_1, T] \end{cases} \quad (39)$$

for $t\in[t_1, T]$ that is for $t\geq t_1$ we have

$$y^*(t)=y_0e^{-at}+be^{-at}\left(\int_0^{t_1} K e^{2at}\,dt+\int_{t_1}^{t} e^{at}\,dt\right),\ t\geq t_1 \quad (40)$$

where $K=-bp(0)>0$.

when $y^*(T)=0$ is imposed one ends up with.

$$y^*(t)=-\frac{b}{a}\left(e^{a(T-t)}-1\right) \quad (41)$$

Which shows that y (t) is négative since $\frac{b}{a}$ is positive and $e^{a(T-t)}>1$ for $t>0$ and $t\in[t_1,T]$

Remember that équation (41) is valid for $t\geq t_1$ so that we can write :

$$y(t)=-b/a\ \left(e^{a(T-t)}-1\right)$$

for $t = t_1$ (42)

and

$$y(t)>-b/a\ \left(e^{a(T-t_1)}-1\right) \text{ for } t<t_1$$

u0 $t\ \_\ t_1$ (43)

since $y(t)<0$ and $a(T-t)$ is decreasing as as $t\to T$

From all the above results we deduce :

$$g^*=\begin{cases} 1 \quad \text{if } -b/a(e^{a(T-t_1)}-1)\leq y(t)< 0,\ t\geq t_1 \\ -p^*b \text{ if } y(t)<-b/a(e^{a(T-t_1)}-1)<0,\ t<t_1 \end{cases} \quad (44)$$

d.5 Case/

$$g^*=\begin{cases} -p^*b \quad \text{for } t\in[0,t_2] \\ \text{and} \quad \text{when } 0\ p(0)\ 1, T\ \_\ t_2 \\ -1 \quad \text{for } t\in[t_2, T] \end{cases} \quad (45)$$

We then have :

$$g^*=\begin{cases} -p^*b \text{ if } -b/a(e^{a(T-t_2)}-1)<y(t),\ t<t_2 \\ -1 \text{ if } 0<y(t)\leq -b/a(e^{a(T-t_2)}-1),\ t\geq t_2 \end{cases} \quad (46)$$

Combining all cases together We get the following optimal control strategy:

$$g^* = \begin{cases} -p^*b \text{ for } y(t) > -b/a\, e^{a(T-t_2)} - 1 > 0 \\ -1 \text{ for } 0 < y(t) \leq -b/a\, e^{a(T-t_2)} - 1 \\ -p^*b \text{ for } y(t) \leq -b/a\, e^{a(T-t_2)} - 1 < 0 \\ +1 \text{ for } -b/a\, e^{a(T-t_1)} - 1 \leq y < 0 \end{cases} \quad (47)$$

CONCLUSION

The above results,and more specifically expressions (44) and(46),show the strong dependence of the optimal policy on the choice of the final time. Depending on the optimal policy can be totally different .In this later case the solution is given in a closed-loop (feedback) form as $g^*(t)$is now a function of the state variable $y^*(t)$, which can be very useful for économic stabilization purposes .

The optimal stretegy(47)tells us,also, about the trade off between the stabilization time and the stabilization control. For a minimum time stabilization policy, we need to apply a control(a bang-bang control) of the "strongest" possible value(here either $g^* = -1$ or $g^* = +1$) throughout the stabilization horizon, while if we are willing to allow more time to the économy to return to its initial équilibrium we need apply,as strategy(4) show, the control $g^*(t) = 1$, for instance, during part of the stabilization time only (from t_1 to T) and the control $g^*(t) = -p^*b$ 1 for the rest of the the time . The value of the optimal time T as well as swetching times $t_1$and t_2 can be derived, by making use of équation(22) and of the boundary condition (T) = 0 . Another difference with the prevailingapproach, is that the assumption of tion of a magnitude constraint on the control variable gives the possibility of corner, just as well as, interior solution. This control constraint,that does exist in the real world, has also an effect on thecontrollability* of the system; this is the reason for the assumption made after équation (13) .

REFERENCES

1- Athans M.and Falb,PLL(1966),Optimal Control, An Introduction to the theory and its application, New-York: Mc Graw-Hill Book. Co.

2- Pontryagin L.S., et al.,(1962):The Mathematical Theory of Optimal Processes, New-York wiley.